Some remarks on relativistic zero-mass wave equations and supersymmetry

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Abstract

We study several formulations of zero-mass relativistic equations, stressing similarities between different frameworks. It is shown that all these massless wave equations have fermionic as well as bosonic solutions.

1   Introduction

Observation that relativistic zero-mass equations may have fermionic as well as bosonic solutions has a long history [1, 2, 3, 4, 5]. More exactly, it was shown that some zero-mas equations, for example the massless Dirac equation, can describe fermionic as well as bosonic states. This property, called the Fermi-Bose duality, was further studied for the massive Dirac equation [6, 7], see also [8, 9] where a notion of supersymmetry was invoked. The most straightforward demonstration of the Fermi-Bose duality in the massless case was given in [5].

It is well known that zero-mass equations, e.g. Maxwell equations or the Weyl neutrino equation, can be cast in several forms. In this note we study relationships between three different formulations leading to these equations. In the next Section we discuss spinor approach to zero-mass equations [5], then in Section 3 a relativistic equation for a massless particle, involving 4×4 matrix representation of the Pauli algebra with a wavefunction with one zero component [10, 11, 12, 13], is described and in Section 4 another approach to the Maxwell equations, based on the massless Dirac equation [3], is analysed. All these approaches are more or less directly related to the Riemann–Silberstein vector [4, 5]. In Section 5 we study generalized Maxwell-like solutions of these equations and in the last Section we discuss the obtained results.

2 Spinor formulation of massless wave equations

All relativistic massless equations can be uniformly written as [5]:

$$\partial_t \varphi_{AB...W} = -c \left(\sigma \cdot \nabla\right)_A^Z \varphi_{ZB...W}, \tag{1}$$

where $\sigma = \left[\sigma^1, \sigma^2, \sigma^3\right]$ are the Pauli matrices.

Equation (1) for first -rank spinor φ_A and second-rank symmetric spinor φ_{AB} leads to:

$$\partial_t \varphi_A = -c \left(\sigma \cdot \nabla\right)_A^Z \varphi_Z, \tag{2}$$

$$\partial_t \varphi_{AB} = -c \left(\sigma \cdot \nabla\right)_A^Z \varphi_{ZB} \quad \left(\varphi_{AB} = \varphi_{BA}\right). \tag{3}$$

Equation (2) is the Weyl neutrino equation while (3) is equivalent to the Maxwell equations. To prove the equivalence, the authors of Ref. ([5]) employed to their Eqn. (12.3) well known relation between φ_{AB} and the Riemann-Silberstein vector $\mathbf{F}_+ = \mathbf{E} + ic\mathbf{B}$:

$$F_+^1 = \phi_{11} - \phi_{00}, \ F_+^2 = -i\left(\phi_{11} + \phi_{00}\right), \ F_+^3 = 2\phi_{01}, \tag{4}$$

where $\phi_{01} = \phi_{10}$.

Equation (3) can be written in matrix form as (cf. Eqn (12.3) of [5]):

$$\partial_0 \Psi = -c\mathbf{\Sigma} \cdot \nabla\, \Psi, \tag{5}$$

where

$$\Sigma^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \Sigma^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \ \Sigma^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
$$\tag{6}$$

Equation (5) has Maxwell as well as neutrino solutions. Indeed, if we choose in (5) $\Psi_M\left(x\right) = \left(\phi_{00}, \ \phi_{01}, \ \phi_{10}, \ \phi_{11}\right)^T$, $\phi_{01} = \phi_{10}$ then Eqn. (3) is recovered while for $\Psi_n\left(x\right) = \varphi \otimes \xi\left(x\right)$ ($\otimes$ denotes the Kronecker product) with constant spinor φ we get $\varphi \otimes \sigma^0 \partial_0 \xi\left(x\right) = -c\varphi \otimes \left(\sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3\right) \xi\left(x\right)$ where we used the obvious representation of the matrices Σ^i:

$$\Sigma^1 = -\sigma^0 \otimes \sigma^1, \ \Sigma^2 = -\sigma^0 \otimes \sigma^2, \ \Sigma^3 = -\sigma^0 \otimes \sigma^3, \tag{7}$$

where $\sigma^0 = 1_2$ is a 2×2 unit matrix (matrices Σ^i are closely related to the Pauli matrices σ_i what is obvious from (7) and, moreover, there are only three such anticommuting matrices).

Moreover, there are also three matrices, $S^1 = \sigma^1 \otimes \sigma^0$, $S^2 = \sigma^2 \otimes \sigma^0$, $S^3 = \sigma^3 \otimes \sigma^0$, commuting with Σ^i's. This result can be used to project – by application of operators $\frac{1}{2}\left(1_4 \pm S^3\right)$ – Eqn. (5) and wavefunction $\Psi_n\left(x\right) = \varphi \otimes \xi\left(x\right)$ onto neutrino subsolution $\xi\left(x\right)$ (note that $1_4 \equiv \sigma^0 \otimes \sigma^0$ is a 4×4 unit matrix) to obtain the Weyl neutrino equation

$$\sigma^0 \partial_0 \, \xi\left(x\right) = -c \left(\sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3\right) \xi\left(x\right), \tag{8}$$

equivalent to (2).

On the other hand, the wave function $\Psi_M = (\phi_{00}, \phi_{01}, \phi_{10}, \phi_{11})^T$, $\phi_{01} = \phi_{10}$, is an eigenstate of the projection operator R:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

i.e. $R\Psi_M = \Psi_M$ and thus we can write the Maxwell subsolution as:

$$\partial_0 R\Psi_M = -c\mathbf{\Sigma} \cdot \nabla R\Psi_M. \tag{10}$$

From Eqn. (10) two equations follow:

$$\partial_0 R\Psi_M = -c\left(R\mathbf{\Sigma}R\right) \cdot \nabla R\Psi_M, \tag{11}$$
$$0 = -c\left(1_4 - R\right)\mathbf{\Sigma} \cdot \nabla R\Psi_M, \tag{12}$$

which correspond to separation of the Maxwell equations (equation (12) is equivalent to Eqn. (12.4) of Ref. [5]).

3 Majorana-Oppenheimer formalism and massless wave equations

According to Majorana and Oppenheimer the Maxwell equations can be written in the following form (cf. [4] and references therein):

$$\partial_0 \tilde{\Psi} = -c\tilde{\mathbf{\Sigma}} \cdot \nabla \tilde{\Psi}, \tag{13}$$

where

$$\tilde{\Sigma}^1 = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tilde{\Sigma}^2 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tilde{\Sigma}^3 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Matrices $\tilde{\Sigma}_i$ can be written as:

$$\tilde{\Sigma}^1 = \sigma^2 \otimes \sigma^3, \ \tilde{\Sigma}^2 = \sigma^0 \otimes \sigma^2, \ \tilde{\Sigma}^3 = \sigma^2 \otimes \sigma^1. \tag{15}$$

It follows that matrices $\tilde{\Sigma}^i$'s are another 4×4 representation of the Pauli matrices, analogous to representation (7). More exactly, there are only three such anticommuting matrices and there are also three matrices, $\tilde{S}^1 = \sigma^2 \otimes \sigma^0$, $\tilde{S}^2 = \sigma^3 \otimes \sigma^2$, $\tilde{S}^3 = \sigma^1 \otimes \sigma^2$, commuting with all $\tilde{\Sigma}^i$'s (matrices $\tilde{\Sigma}^i$, $\tilde{S}^j$ are proportional to matrices α^i, β^j introduced in [4]).

Eqn. (13) has fermionic as well as bosonic solution with close analogy to (5). Indeed, choosing $\tilde{\Psi}$ as $\tilde{\Psi}_n(x) = \varphi \otimes \eta(x)$ where $\varphi = (-i,\ 1)^T$ is eigenvector of σ^2, $\sigma^2\varphi = +\varphi$, we arrive at equation:

$$\left(\sigma^0\partial_0 + c\sigma^3\partial_1 + c\sigma^2\partial_2 + c\sigma^1\partial_3\right)\eta(x) = 0, \tag{16}$$

which can be transformed by application of the unitary operator $U = \frac{1}{\sqrt{2}}\sigma^2\left(\sigma^1 + \sigma^3\right)$ to the neutino equation (this property of Eqn. (13) hasn't been noticed before):

$$\left(\sigma^0\partial_0 - c\sigma^1\partial_1 - c\sigma^2\partial_2 - c\sigma^3\partial_3\right)U\eta(x) = 0. \tag{17}$$

On the other hand, for $\tilde{\Psi}_M(x) = \left(0,\ F_+^1,\ F_+^2,\ F_+^3\right)$, where $\mathbf{F}_+$ is again the Riemann-Silberstein vector, Eqn. (13) is equivalent to the Maxwell equations [13, 4]. We can thus write:

$$\partial_0\tilde{R}\tilde{\Psi}_M = -c\tilde{\mathbf{\Sigma}} \cdot \nabla\,\tilde{R}\tilde{\Psi}_M, \tag{18}$$

where

$$\tilde{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

The following equations

$$\partial_0\tilde{R}\tilde{\Psi}_M = -c\left(P\tilde{\mathbf{\Sigma}}P\right) \cdot \nabla\,\tilde{R}\tilde{\Psi}_M, \tag{20}$$

$$0 = -c\left(1_4 - \tilde{R}\right)\tilde{\mathbf{\Sigma}} \cdot \nabla\,\tilde{R}\tilde{\Psi}_M, \tag{21}$$

are equivalent to Eqns. (11), (12).

4 Four-component massless Dirac equation: neutrino and Maxwell solutions

We shall discuss now fermionic and bosonic subsolutions of the massless Dirac equation:

$$i\gamma^\mu\partial_\mu\Psi = 0, \tag{22}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $x^0 = ct$. Projection operators $Q_\pm = \frac{1}{2}\left(1_4 \pm \gamma^5\right)$, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, in spinor representation of the Dirac matrices

$$\gamma_0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix},\ \gamma = \begin{pmatrix} 0_2 & -\sigma \\ \sigma & 0_2 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} \sigma^0 & 0_2 \\ 0_2 & -\sigma^0 \end{pmatrix}, \tag{23}$$

where 0_2 is a 2×2 zero matrix, split (22) into the Weyl neutrino/antineutrino equations:

$$i\partial_t\xi = -c\sigma \cdot \nabla\xi, \tag{24}$$

$$i\partial_t\eta = +c\sigma \cdot \nabla\eta. \tag{25}$$

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The massless Dirac equation has also bosonic solutions. Simulik and Krivsky demonstrated [3] that the following substitution:

$$\Psi_M = (iE_3, \ iE_1 - E_2, \ -cB_3, \ -icB_2 - cB_1)^T \,, \qquad (26)$$

introduced into the Dirac equation (22) converts it for standard representation of the Dirac matrices (27)

$$\gamma_0 = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}, \ \gamma = \begin{pmatrix} 0_2 & \sigma \\ -\sigma & 0_2 \end{pmatrix}, \ \gamma_5 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \qquad (27)$$

into the set of Maxwell equations (note that in [3] convention $\hbar = c = 1$ was used). Let us also note that $\gamma^5 \Psi_M$, where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, substituted for Ψ in (22) leads to the same Maxwell equations.

5 Generalized Maxwell-like equations

It has been found that zero-mass Dirac equation leads to generalized Maxwell equations. Indeed, substituting $\Psi = \Theta_M$ in (22),

$$\Theta_M = \left(iE^3 - B^0, \ iE^1 - E^2, \ iE^0 - B^3, \ -iB^2 - B^1\right)^T \,, \qquad (28)$$

we get:

$$\begin{aligned}
c\nabla \cdot \mathbf{E} &= -\partial_t E_0 \\
c\nabla \cdot \mathbf{B} &= -\partial_t B_0 \\
\tfrac{1}{c}\partial_t \mathbf{E} - c\nabla \times \mathbf{B} &= -\nabla E_0 \\
\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= -c\nabla B_0
\end{aligned} \qquad (29)$$

i.e. Maxwell equations with gradient-type sources [3].

It is interesting that virtually the same equations arise from Eqns. (10), (18) if we remove some restrictions imposed on the appropriate wavefunctions. Indeed, substituting $\Psi = \Phi_M$ into (10) with

$$\Phi_M = \left(-\tfrac{1}{2}\zeta^{1\dot{2}}, \tfrac{1}{2}\zeta^{1\dot{1}}, -\tfrac{1}{2}\zeta^{2\dot{2}}, \tfrac{1}{2}\zeta^{2\dot{1}}\right)^T$$
$$\zeta^{1\dot{1}} = F_+^3 + F_+^0, \ \zeta^{1\dot{2}} = F_+^1 - iF_+^2, \ \zeta^{2\dot{1}} = F_+^1 + iF_+^2, \ \zeta^{2\dot{2}} = -F_+^3 + F_+^0 \qquad (30)$$

we get Eqns. (29) while for $\Psi = \tilde{\Phi}_M$ inserted into (18) where

$$\tilde{\Phi}_M = \left(F_+^0, \ F_+^1, \ F_+^2, \ F_+^3\right)^T \,, \qquad (31)$$

we obtain Eqns. (29) with E^0 and B^0 interchanged. In all three cases we recover the Maxwell equations demanding that $E^0 = B^0 = 0$.

6   Discussion

We have studied several formulations of zero-mass relativistic wave equations. It turns out that all considered equations, (5), (13), (22), have fermionic (Weyl neutrino) as well as bosonic (Maxwell) solutions. Equations (5), (13) have very similar structure, however existence of fermionic solutions in the latter case wasn't noticed before. Note that the matrices Σ_i's (as well as $\tilde{\Sigma}_i$'s) are a 4×4 representation of the Pauli algebra and the corresponding equations are analogues of the Weyl equation. It is interesting that equations (5), (13), (22) lead to virtually identical generalized Maxwell-like equations, discovered for Eqn. (22) in [3]. These Maxwell-like equations can be related to the Dirac equation with non-zero mass (however for the case of a static scalar potential only) thus extending the Fermi-Bose duality to the massive case [14] (see also earlier ideas exposed in Refs. [15, 16]). The Fermi-Bose duality, referred to as supersymmetry, was also studied in Refs. [8, 9], placing earlier results [17, 18] in a new context.

Our work is related to other studies of massless equations. It was shown that square of the Dirac operator is supersymmetric, containing fermion and boson sectors, and this was used to study the zero-mass Dirac equation in the interacting case in flat [19] and curved space [20]. There is also a very interesting analogy between Fermi-Bose duality discussed above and duality invariance characteristic for the Maxwell field – all free bosonic and fermionic gauge fields are invariant with respect to duality transformation [21].

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